IIT - JEE ADVANCED - 2012

PAPER-2 [Code – 8]

PART - III: MATHEMATICS

SECTION I : Single Correct Answer Type

This section contains **8 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE is correct.**

41. Let a_1, a_2, a_3, ….. be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$

(A) 22
(B) 23
(C) 24
(D) 25

Sol. **(D)**

a_1, a_2, a_3, are in H.P.

$\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are in A.P.

$\Rightarrow \frac{1}{a_n} = \frac{1}{a_1} + (n-1)d < 0$, where $\frac{\frac{1}{25} - \frac{5}{25}}{19} = d = \left(\frac{-4}{9\times 25}\right)$

$\Rightarrow \frac{1}{5} + (n-1)\left(\frac{-4}{19\times 25}\right) < 0$

$\frac{4(n-1)}{19\times 5} > 1$

$n - 1 > \frac{19\times 5}{4}$

$n > \frac{19\times 5}{4} + 1 \Rightarrow n \geq 25.$

42. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is

(A) $5x - 11y + z = 17$
(B) $\sqrt{2}x + y = 3\sqrt{2} - 1$
(C) $x + y + z = \sqrt{3}$
(D) $x - \sqrt{2}y = 1 - \sqrt{2}$

Sol. **(A)**

Equation of required plane is

$P \equiv (x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$

$\Rightarrow (1 + \lambda)x + (2 - \lambda)y + (3 + \lambda)z - (2 + 3\lambda) = 0$

Its distance from (3, 1, –1) is $\frac{2}{\sqrt{3}}$

$\Rightarrow \frac{2}{\sqrt{3}} = \frac{|3(1+\lambda) + (2-\lambda) - (3+\lambda) - (2+3\lambda)|}{\sqrt{(\lambda+1)^2 + (2-\lambda)^2 + (3+\lambda)^2}}$

$= \frac{4}{3} = \frac{(-2\lambda)^2}{3\lambda^2 + 4\lambda + 14} \Rightarrow 3\lambda^2 + 4\lambda + 14 = 3\lambda^2$

$\Rightarrow \lambda = -\frac{7}{2} \Rightarrow -\frac{5}{2}x + \frac{11}{2}y - \frac{z}{2} + \frac{17}{2} = 0$

$-5x + 11y - z + 17 = 0.$

43. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b, and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$

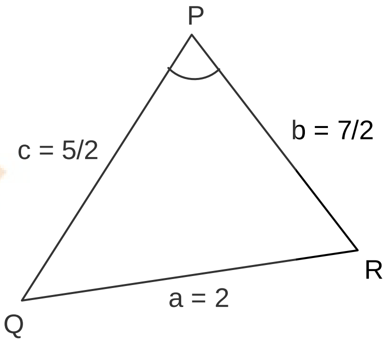
(C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

Sol. **(C)**

$$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)}$$

$$= \frac{((s-b)(s-c))^2}{\Delta^2} = \frac{\left(\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$



44. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}| = \sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$, then a possible value of $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$ is

(A) 0 (B) 3

(C) 4 (D) 8

Sol. **(C)**

$\vec{a}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right) = \left(2\hat{i}+3\hat{j}+4\hat{k}\right)\times\vec{b}$

$\left(\vec{a}+\vec{b}\right)\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right) = \vec{0}$

$\Rightarrow \vec{a}+\vec{b} = \pm\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$ (as $\left|\vec{a}+\vec{b}\right| = \sqrt{29}$)

$\Rightarrow \left(\vec{a}+\vec{b}\right)\cdot\left(-7\hat{i}+2\hat{j}+3\hat{k}\right)$

$= \pm(-14 + 6 + 12) = \pm 4.$

45. If P is a 3 × 3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3 × 3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) $PX = X$

(C) $PX = 2X$ (D) $PX = -X$

Sol. **(D)**

Give $P^T = 2P + I$

$\Rightarrow P = 2P^T + I = 2(2P + I) + I$

$\Rightarrow P + I = 0$

$\Rightarrow PX + X = 0$

$PX = -X.$

46. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a}-1)x^2+(\sqrt{1+a}-1)x+(\sqrt[6]{1+a}-1)=0$ where $a > -1$. Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are

(A) $-\frac{5}{2}$ and 1

(B) $-\frac{1}{2}$ and -1

(C) $-\frac{7}{2}$ and 2

(D) $-\frac{9}{2}$ and 3

Sol. **(B)**

Let $1 + a = y$

$\Rightarrow (y^{1/3} - 1)x^2 + (y^{1/2} - 1)x + y^{1/6} - 1 = 0$

$\Rightarrow \left(\frac{y^{1/3}-1}{y-1}\right)x^2+\left(\frac{y^{1/2}-1}{y-1}\right)x+\frac{y^{1/6}-1}{y-1}=0$

Now taking $\lim_{y\to 1}$ on both the sides

$\Rightarrow \frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{6}=0$

$\Rightarrow 2x^2 + 3x + 1 = 0$

$x = -1, -\frac{1}{2}$.

47. Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5, and 6, are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is

(A) $\frac{91}{216}$

(B) $\frac{108}{216}$

(C) $\frac{125}{216}$

(D) $\frac{127}{216}$

Sol. **(A)**

Required probability $= 1 - \frac{6\cdot 5^3}{6^4} = 1-\frac{125}{216}=\frac{91}{216}$.

48. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is

(A) 0

(B) $\frac{\pi^2}{2}-4$

(C) $\frac{\pi^2}{2}+4$

(D) $\frac{\pi^2}{2}$

Sol. **(B)**

$\int_{-\pi/2}^{\pi/2}\left\{x^2+\ln\left(\frac{\pi+x}{\pi-x}\right)\right\}\cos x dx$

$= \int_{-\pi/2}^{\pi/2} x^2\cos x dx + \int_{-\pi/2}^{\pi/2}\ln\left(\frac{\pi+x}{\pi-x}\right)\cos x dx$

$= 2\int_{0}^{\pi/2} x^2\cos x dx$

$= 2\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}$

$$= 2\left[\frac{\pi^2}{4} - 2\right] = \frac{\pi^2}{2} - 4.$$

SECTION II : Paragraph Type

This section contains **6 multiple choice questions** relating to three paragraphs with **two questions on each paragraph.** Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE is correct.**

Paragraph for Questions 49 and 50

A tangent *PT* is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line *L*, perpendicular to *PT* is a tangent to the circle $(x-3)^2 + y^2 = 1$.

49. A possible equation of *L* is

(A) $x - \sqrt{3}y = 1$ (B) $x + \sqrt{3}y = 1$

(C) $x - \sqrt{3}y = -1$ (D) $x + \sqrt{3}y = 5$

Sol. **(A)**

Equation of tangent at $P\left(\sqrt{3}, 1\right)$

$\sqrt{3}x + y = 4$

Slope of line perpendicular to above tangent is $\frac{1}{\sqrt{3}}$

So equation of tangents with slope $\frac{1}{\sqrt{3}}$ to $(x-3)^2 + y^2 = 1$ will be

$y = \frac{1}{\sqrt{3}}(x-3) \pm 1\sqrt{1+\frac{1}{3}}$

$\sqrt{3}\,y = x - 3 \pm (2)$

$\sqrt{3}\,y = x - 1$ or $\sqrt{3}\,y = x - 5$.

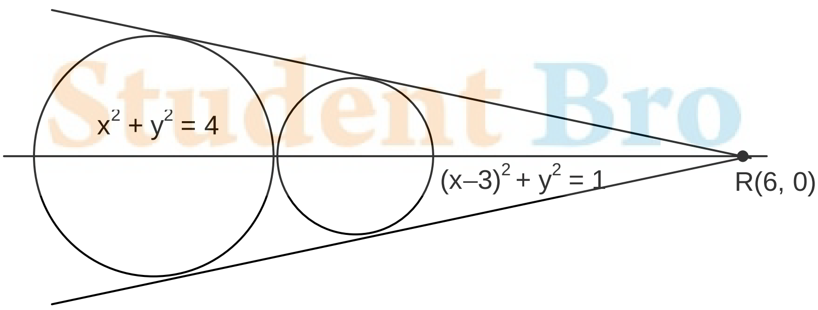
50. A common tangent of the two circles is

(A) $x = 4$ (B) $y = 2$

(C) $x + \sqrt{3}y = 4$ (D) $x + 2\sqrt{2}y = 6$

Sol. **(D)**

Point of intersection of direct common tangents is (6, 0)



so let the equation of common tangent be

$y - 0 = m(x - 6)$

as it touches $x^2 + y^2 = 4$

$\Rightarrow \left|\frac{0 - 0 + 6m}{\sqrt{1+m^2}}\right| = 2$

$9m^2 = 1 + m^2$

$m = \pm\dfrac{1}{2\sqrt{2}}$

So equation of common tangent

$y = \dfrac{1}{2\sqrt{2}}(x-6)$, $y = -\dfrac{1}{2\sqrt{2}}(x-6)$ and also $x = 2$

Paragraph for Questions 51 and 52

Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in IR$, and let $g(x) = \int_1^x \left(\dfrac{2(t-1)}{t+1} - \ln t\right) f(t)\, dt$ for all $x \in (1, \infty)$.

51. Consider the statements:

P : There exists some $x \in$ □ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in$ □ such that $2f(x) + 1 = 2x(1 + x)$

Then

(A) both **P** and **Q** are true
(B) **P** is true and **Q** is false
(C) **P** is false and **Q** is true
(D) both **P** and **Q** are false

Sol. **(C)**

$f(x) = (1-x)^2\sin^2 x + x^2 \qquad \forall\, x \in R$

$g(x) = \int_1^x \left(\dfrac{2(t-1)}{t+1} - \ln t\right) f(t)\, dt \qquad \forall\, x \in (1, \infty)$

For statement P :

$f(x) + 2x = 2(1 + x^2) \qquad \ldots(i)$

$(1-x)^2\sin^2 x + x^2 + 2x = 2 + 2x^2$

$(1-x)^2\sin^2 x = x^2 - 2x + 2 = (x-1)^2 + 1$

$(1-x)^2 (\sin^2 x - 1) = 1$

$-(1-x)^2\cos^2 x = 1$

$(1-x)^2.\cos^2 x = -1$

So equation (i) will not have real solution

So, P is wrong.

For statement Q :

$2(1-x)^2\sin^2 x + 2x^2 + 1 = 2x + 2x^2 \qquad \ldots(ii)$

$2(1-x)^2\sin^2 x = 2x - 1$

$2\sin^2 x = \dfrac{2x-1}{(1-x)^2}$ Let $h(x) = \dfrac{2x-1}{(1-x)^2} - 2\sin^2 x$

Clearly $h(0) = -$ve, $\lim_{x\to 1^-} h(x) = +\infty$

So by IVT, equation (ii) will have solution.

So, Q is correct.

52. Which of the following is true?

(A) g is increasing on $(1, \infty)$
(B) g is decreasing on $(1, \infty)$
(C) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
(D) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

Sol. **(B)**

$g'(x) = \left(\dfrac{2(x-1)}{x+1} - \ln x\right) f(x)$. For $x \in (1, \infty)$, $f(x) > 0$

Let $h(x) = \left(\dfrac{2(x-1)}{x+1} - \ln x\right) \Rightarrow h'(x) = \left(\dfrac{4}{(x+1)^2} - \dfrac{1}{x}\right) = \dfrac{-(x-1)^2}{(x+1)^2 x} < 0$

Also $h(1) = 0$ so, $h(x) < 0 \quad \forall\, x > 1$

$\Rightarrow g(x)$ is decreasing on $(1, \infty)$.

Paragraph for Questions 53 and 54

Let a_n denote the number of all n–digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n–digit integers ending with digit 1 and c_n = the number of such n–digit integers ending with digit 0.

53. The value of b_6 is

(A) 7 (B) 8
(C) 9 (D) 11

Sol. **(B)**

$a_n = b_n + c_n$

$b_n = a_{n-1}$

$c_n = a_{n-2} \Rightarrow a_n = a_{n-1} + a_{n-2}$

As $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8 \Rightarrow b_6 = 8$.

54. Which of the following is correct?

(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$
(C) $b_{17} \neq b_{16} + c_{16}$ (D) $a_{17} = c_{17} + b_{16}$

Sol. **(A)**

As $a_n = a_{n-1} + a_{n-2}$

for $n = 17$

$\Rightarrow a_{17} = a_{16} + a_{15}$.

SECTION III : Multiple Correct Answer(s) Type

This section contains **6 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which **ONE or MORE are correct.**

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55. For every integer n, let a_n and b_n be real numbers. Let function f : IR → IR be given by

$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases}$, for all integers n. If f is continuous, then which of the following hold(s) for all n?

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$
(C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

Sol. **(B, D)**

At x = 2n

L.H.L. $= \lim_{h\to 0}\left(b_n + \cos \pi(2n-h)\right) = b_n + 1$

R.H.L. $= \lim_{h\to 0}\left(a_n + \sin \pi(2n+h)\right) = a_n$

$f(2n) = a_n$

For continuity $b_n + 1 = a_n$

At $x = 2n + 1$

L.H.L $= \lim_{h\to 0}\left(a_n + \sin \pi(2n+1-h)\right) = a_n$

R.H.L $= \lim_{h\to 0}\left(b_{n+1} + \cos\left(\pi(2n+1-h)\right)\right) = b_{n+1} - 1$

$f(2n + 1) = a_n$

For continuity

$a_n = b_{n+1} - 1$

$a_{n-1} - b_n = -1$.

56. If the straight lines $\frac{x-1}{2}=\frac{y+1}{k}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are)

(A) $y + 2z = -1$ (B) $y + z = -1$
(C) $y - z = -1$ (D) $y - 2z = -1$

Sol. **(B, C)**

For given lines to be coplanar, we get

$$\begin{vmatrix} 2 & k & 2 \\ 5 & 2 & k \\ 2 & 0 & 0 \end{vmatrix} = 0 \Rightarrow k^2 = 4, k = \pm 2$$

For $k = 2$, obviously the plane $y + 1 = z$ is common in both lines
For $k = -2$, family of plane containing first line is $x + y + \lambda(x - z - 1) = 0$.
Point $(-1, -1, 0)$ must satisfy it
$-2 + \lambda(-2) = 0 \Rightarrow \lambda = -1$
$\Rightarrow y + z + 1 = 0$.

57. If the adjoint of a 3 × 3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are)

(A) –2 (B) –1
(C) 1 (D) 2

Sol. **(A, D)**

$|\text{Adj } P| = |P|^2$ as $(|\text{Adj }(P)| = |P|^{n-1})$
Since $|\text{Adj } P| = 1(3 - 7) - 4(6 - 7) + 4(2 - 1)$
$= 4$
$|P| = 2$ or -2.

58. Let $f : (-1, 1) \to \text{IR}$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are)

(A) $1-\sqrt{\frac{3}{2}}$ (B) $1+\sqrt{\frac{3}{2}}$
(C) $1-\sqrt{\frac{2}{3}}$ (D) $1+\sqrt{\frac{2}{3}}$

Sol. **(A, B)**

For $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Let $\cos 4\theta = 1/3$

$\Rightarrow \cos 2\theta = \pm\sqrt{\frac{1+\cos 4\theta}{2}} = \pm\sqrt{\frac{2}{3}}$

$$f\left(\frac{1}{3}\right) = \frac{2}{2-\sec^2\theta} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = 1 + \frac{1}{\cos 2\theta}$$

$$f\left(\frac{1}{3}\right) = 1 - \sqrt{\frac{3}{2}} \text{ or } 1 + \sqrt{\frac{3}{2}}.$$

59. Let X and Y be two events such that $P(X|Y) = \frac{1}{2}$, $P(Y|X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is (are) correct?

(A) $P(X \cup Y) = \frac{2}{3}$

(B) X and Y are independent

(C) X and Y are not independent

(D) $P(X^C \cap Y) = \frac{1}{3}$

Sol. **(A, B)**

$P\left(\frac{X}{Y}\right) = \frac{P(X \cap Y)}{P(Y)} = \frac{1}{2}$ and $\frac{P(X \cap Y)}{P(X)} = \frac{1}{3}$

$P(X \cap Y) = \frac{1}{6} \Rightarrow$ P (Y) = $\frac{1}{3}$ and P (X) = $\frac{1}{2}$

Clearly, X and Y are independent

Also, $P(X \cup Y) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$.

60. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then

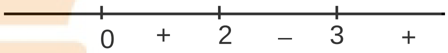
(A) f has a local maximum at $x = 2$

(B) f is decreasing on (2, 3)

(C) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$

(D) f has a local minimum at $x = 3$

Sol. **(A, B, C, D)**



$f'(x) = e^{x^2}(x-2)(x-3)$

Clearly, maxima at $x = 2$, minima at $x = 3$ and decreasing in $x \in (2, 3)$.

$f'(x) = 0$ for $x = 2$ and $x = 3$ (Rolle's theorem)

so there exist $c \in (2, 3)$ for which

$f''(c) = 0$.